Example (2):- The Complex linear space C of all Complex numbers is a Banach space w.r.t. the norm defined by,

$$\|z\| = |z| \text{ for all } z \in C.$$

Verification: Clearly, $\|z\| = |z| \geq 0$ $\|z\| = 0$ iff $|z| = 0$ iff $z = 0$.

Also, $\|\alpha z\| = |\alpha \cdot z| = |\alpha| \cdot |z\| = |\alpha| \cdot \|z\|$ for all $\alpha, z \in C$.

Finally, $\|z + w\| = |z + w| \leq + |w\| = \|z\| + \|w\|$ for all $z, w \in C$.

$\therefore$ C is nls the metric d generated by the norm is given by $d(z, w) = \|z - w\| = |z - w|$ for all $z, w \in C$.

We show that, (C, d) is a Complete metric space.

Let $\{z_n\}$ be any Cauchy sequence in C. then $d(z_m, z_n) = \|z_m - z_n\| = |z_m - z_n| \to 0$ as $m, n \to \alpha$.

Now, let $z_n = x_n + i y_n$ for $n = 1, 2, 3 - - -$ then $|x_m - x_n| \leq |z_m - z_n| \to 0$ as $m, n \to \infty$.

$$|y_m - y_n| \leq |z_m - z_n| \to 0 \text{ as } m, n \to \infty$$

then, both $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences of real numbers. Since $\mathbb{R}$ is complete, there exists $x, y \in \mathbb{R}$ such that $x_n \to x$ & $y_n \to y$ as $n \to \infty$ i.e. $|x_n - x| \to 0$ as $n \to \infty$.

Let $z = x + iy$, then $z \in \mathbb{C}$.

Also, $(z_n, z) = \|z_n - z\| = |z_n - z|$

$= |x_n + iy_n - (x + iy)| = |(x_n - x) + i(y_n - y)|$

$\leq |x_n - x| + |i(y_n - y)|$

$\therefore d(z_n, z) = |x_n - x| + |y_n - y| \to 0$ as $n \to \infty$.

$\therefore \{z_n\}$ converges to $z \in \mathbb{C}$. Hence $(\mathbb{C}, d)$ is a complete metric space ω for all $z, w \in \mathbb{C}$.

$\therefore \mathbb{C}$ is a Banach space.

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QN.→ Define normal operator on a Hilbert Space H. If A is a normal operator on a Complex Hilbert Space H, Prove that $\|A\|^2 = \|A^2\|$.

(Defn) Normal operators:- An operator N on a Hilbert Space H is said to be a normal operater, If it commutes with its adjoint i.e. $NN^* = N^*N$.

We know that an operator T on H is normal $\Leftrightarrow \|T^*x\| = \|Tx\|$ for every $x \in H$.

For every $x \in H$,

$$\|A^2 x\| = \|AAx\| = \|Ay\| \quad [\text{Putting } y = Ax]$$

$$= \|A^* y\| \quad [\because A \text{ is normal}]$$

$$= \|A^* A x\|$$

$$\therefore \|A^2\| = \|A^* A\| \quad \text{—— (1)}$$

Also $\|A^* A\| \leq \|A^*\| \cdot \|A\|$

$$= \|A\| \cdot \|A\| = \|A\|^2 \quad \text{—— (2)}$$

Moreover, $\|Ax\|^2 = (Ax, Ax) = (A^* A x, x)$

$$\leq \|A^* A x\| \cdot \|x\|$$

$$\leq \|A^* A\| \cdot \|x\|^2$$

$$\therefore \|A\|^2 \leq \|A^* A\| \quad \text{—— (3)}$$

From 2 & 3, we have

$$||A^* A|| = ||A||^2 \quad \text{—— } (4)$$

From 1 & 4, we have

$$||A^2|| = ||A||^2 .$$

Q.No → If T is an arbitrary operator on a finite dimensional non-zero space then the eigenvalues of T Constitute a non-empty finite subset of the Complex Plane. Furthermore the number of Points in this set does not exceed the dimension n of the Space H. Prove it.

Proof:- We note that λ is an eigenvalue of T

$\Leftrightarrow$ there exists a non-zero vector x Such that

$\Leftrightarrow$ $(T-\lambda I)x = 0$.

$\Leftrightarrow$ $T-\lambda I$ is Singular

$\Leftrightarrow$ $\det(T-\lambda I) = 0$ where "det" denotes the determinant of the matrix $T-\lambda I$.

Thus the eigenvalues of T are Precisely the distinct roots of equation.

$$\det(T-\lambda I) = 0 \quad \text{——} \quad (1)$$

Which is called the Characteristic equation of T.

If $[\alpha_{ij}]$ be the matrix of T relative to an ordered basis B for H, the Characteristic equation Can be written in the form,

$$\begin{vmatrix} \alpha_{11}-\lambda & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22}-\lambda & \cdots & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn}-\lambda \end{vmatrix} = 0 \quad \text{——} \quad (2)$$

This is a Polynomial equation with Complex Coefficients, of degree n in the Complex variable λ. Such an equation has exactly n Complex roots (i.e. eigenvalues of T) Some of which may be repeated, in which Case there are less than n distinct root.